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BENDING OF A HIGHLY STRETCHED PLATE CONTAINING AN ECCENTRICALLY PLATE-REINFORCED CIRCULAR HOLE

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Abstract—In this paper, we consider the problem of finding the stress distribution in a highly stretched plate containing a circular hole that is eccentrically reinforced by thickening the plate, on one side only, in an annular region concentric with the hole. A solution of the nonlinear Kármán plate equations is obtained that is asymptotically valid for large membrane stresses. We show that, except for a narrow bending boundary layer in the neighbourhood of the boundary between the reinforced area and the rest of the plate, a state of plane stress prevails and the reinforced area undergoes a transverse deflection that brings its middle surface into the plane of the middle surface of the plate.

NOTATION

Note: All barred variables are dimensional variables.

D	$Eh^3/[12(1-\nu^2)]$, flexural stiffness
E	Young's modulus
$(\bar{e}^{rr}, \bar{e}^{r\theta}, \bar{e}^{\theta\theta})$	$P(Eh)^{-1}(e^{rr}, e^{r\theta}, e^{\theta\theta})$, middle surface strains
$\bar{f}$	PR^2f, Airy's stress function
h	thickness of plate
H	thickness of reinforcement
$(\bar{M}^{rr}, \bar{M}^{r\theta}, \bar{M}^{\theta\theta})$	$Ph\gamma^{-1}(M^{rr}, M^{r\theta}, M^{\theta\theta})$, moments
$(\bar{N}^{rr}, \bar{N}^{r\theta}, \bar{N}^{\theta\theta})$	$P(N^{rr}, N^{r\theta}, N^{\theta\theta})$, membrane stress resultants
$\bar{n}(\theta)$	$Pn(\theta) = \bar{N}^{rr}(\theta, R)$, radial stress resultant at the junction of the plate and reinforcement
P	applied load (see Fig. 1)
$(\bar{Q}^r, \bar{Q}^\theta)$	$Ph(R\gamma)^{-1}(Q^r, Q^\theta)$, transverse shear stress resultants
$\bar{Q}_{\text{eff}}$	$Ph(R\gamma)^{-1}Q_{\text{eff}}$, effective transverse shear
R	outer radius of reinforcement
(r, θ, Z)	(Rx, θ, hz), cylindrical polar coordinates
$(\bar{u}, \bar{v})$	$RP(Eh)^{-1}(u, v)$, inplane middle surface displacements
$\bar{w}$	$h\gamma^{-1}w$, transverse displacement
(X, Y, Z)	$(Rx\cos\theta, Rx\sin\theta, hz)$, dimensional cartesian coordinates
α	load ratio (see Fig. 1)
β	1 in the plate and λ in the reinforcement
γ	$[6(1-\nu^2)]^{1/2}$
δ	$(1-\lambda)/(2\lambda)$, dimensionless eccentricity of the middle surfaces
ϵ^2	$D/(Pr^2)$, a small parameter
λ	h/H, plate thickness to reinforcement thickness ratio
ν	Poisson's ratio
ρR	radius of hole
σ	dimensional radial extreme fibre stress in the lower surface of the plate and the upper surface of the reinforcement at $r = R$.

1. INTRODUCTION

The problem of finding the stress distribution in a stretched plate containing a hole that is reinforced by thickening the plate symmetrically about its middle surface in a region surrounding the hole has received considerable attention. For example, Chapter five of the book by Savin[1] gives solutions to a number of problems involving circular and non-circular holes, and Wittrick[2] gives several references to other work. The solutions are all obtained using the theory of plane stress.

If the hole is reinforced unsymmetrically, by thickening the plate on one side only so that the other side remains smooth, then the stretching of the plate is accompanied by bending. The equations appropriate to this situation are the Kármán large deflection plate equations, in which the plane stress equations and the plate bending equations are coupled by the presence of non-linear terms. This is a more difficult problem and fewer results are available.

The problem of a thin plate of infinite extent, containing a circular hole reinforced by uniformly thickening one side of the plate in an annular region concentric with the hole, subjected to an axially symmetric radial stress at infinity, has been solved by Wittrick[2, 3]. In [3] he obtains an asymptotic solution, valid for large stress at infinity. Reference[2] gives a more general discussion of the axially symmetric problem and also contains an asymptotic solution for the case when the added thickness is small.

Hicks had earlier considered both the above problem[4], and the corresponding problem with a compact reinforcing ring eccentrically placed relative to the plate middle surface[5]. However, as Wittrick points out in [2], the validity of Hicks' results is questionable since, although he takes account of the effect of the membrane stresses on the equations of bending moment equilibrium, the nonlinear effect of the transverse deflection on the membrane strains is ignored. Also, in [4], he ignores the effect of the eccentricity of the middle surfaces in the equation of compatibility of radial displacement at the edge of the reinforced area.

So far no one has considered the problem of a plate with an eccentrically reinforced circular hole subject to non-axisymmetric stress at a large distance from the hole, or the case of a noncircular hole. Alzheimer and Davis[6, 7] consider the closely related problem of the unsymmetrical bending of an annular plate where the bending is caused by the tilting, about its diameter, of a rigid disc inserted in the annulus, the plate being fully built in at the disc and at the outer radius. In [7] the plate is prestressed by a large axisymmetric stress applied at its outer edge. They use a singular perturbation procedure similar to that used by Wittrick[3] to obtain a solution which they then compare with the exact solution of their equations. These solutions are open to the same criticism as those obtained by Hicks in that they ignore the effect of the transverse displacement on the membrane strains. However, when the prestress is very large, this effect is second order, as is shown in [3] and as will also be shown here, and in fact Alzheimer and Davis state this as an assumption in their introduction to the paper of reference[7].

In this paper we look at the Wittrick problem when the plate is highly stretched by a large biaxial stress system at infinity. That is, referred to cartesian coordinates X, Y and the stress resultants at infinity are (see Fig. 1)

$$\alpha\bar{N}^{XX} = \bar{N}^{XY} = \alpha P, \qquad \bar{N}^{XY} = 0, \qquad 0 \leq \alpha \leq 1. \tag{1}$$

The solution obtained is asymptotically valid for large load P, and small bending stiffness. Notice that when the load ratio $\alpha = 1$ the problem reduces to that solved by Wittrick[3], and when $\alpha = 0$ the plate is in a state of uniaxial stress at infinity.

As will be shown, this is a typical singular perturbation problem that can be solved by a fairly straightforward application of the method of matched asymptotic expansion as described for example by Cole[8] and Nayfeh[9]. The result is that, except for a narrow bending boundary layer in the neighbourhood of the junction between the plate and the reinforced area, a state of plane stress prevails and the reinforcing plate undergoes a transverse deflection that brings its

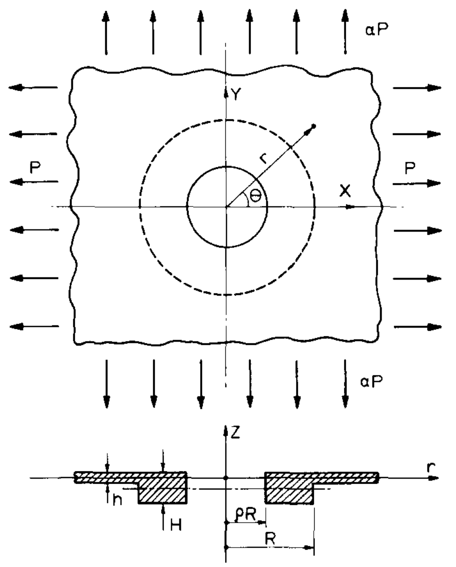

Fig. 1. Eccentrically reinforced hole in a plate loaded by biaxial stresses far from the hole.

middle surface into the plane of the middle surface of the rest of the plate (see Fig. 2). The nature of the solution also suggests that this result will apply to any highly stretched plate containing a hole of arbitrary shape that is reinforced by thickening the plate eccentrically in a region surrounding the hole, providing that the junction between the plate and the reinforced area is in a state of tension.

Wittrick[3], was able to evaluate all the terms in the boundary layer expansion mainly because, in the axisymmetric case, the equations can be reduced to a pair of coupled, nonlinear, second order ordinary differential equations. Here we must deal with fourth order partial differential equations and, for this reason, only the leading term in the boundary layer expansions will be obtained. It is clear that with sufficient labour the higher order terms could be evaluated. In the next section the Kármán large deflection plate equations are stated in the dimensionless form that is appropriate to this problem and the boundary conditions are formulated. Sections 3 and 4 outline the application of the method of matched asymptotic expansions to these equations and may be skipped if you are prepared to take the validity of the mathematical manipulations on trust and are mainly interested in the results. Finally, Section 5 summarizes the main results in terms of the physical, dimensional quantities of the problem and discusses the conjecture made in the previous paragraph.

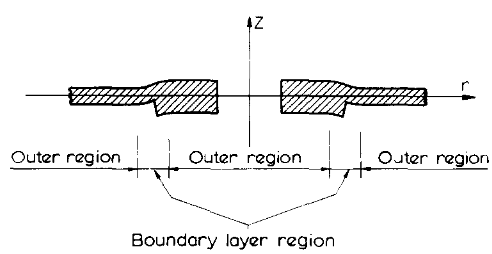

Fig. 2. Sketch showing the deformation of the plate when the load becomes very large.

2. FORMULATION OF THE PROBLEM

Consider an infinite plate of thickness h subject, at infinity, to the biaxial stress state specified by equations (1). The plate contains a hole of radius ρR which is reinforced by increasing its thickness, on one side only, to H in an annular region with outer radius R (Fig. 1). We shall assume that the thickness of both sections of the plate is sufficiently small compared to the size of the annulus, $(1-\rho)R$, for the Kármán equations for large deflection of plates to apply. For brevity, we shall refer to the annular reinforcing plate as the "reinforcement" and to the remaining plate ($r \geq R$) simply as the "plate".

2.1 *The nondimensional equations*

Since the Kármán equations for large deflection of plates are well known (see, for example, Fung[10]) we set them down immediately in the dimensionless form that is most useful for the present analysis. They are

$$\epsilon^2\nabla^4 w = \beta^3 L(w,f), \qquad \nabla^4 f = -\epsilon^2\beta^{-1}L(w,w), \tag{2a, b}$$

where

$$\nabla^4 \equiv (\nabla^2)^2 \equiv \left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)^2, \tag{3a}$$

$$L(A,B) \equiv \frac{\partial^2 A}{\partial x^2}\left(\frac{1}{x^2}\frac{\partial^2 B}{\partial\theta^2}+\frac{1}{x}\frac{\partial B}{\partial x}\right)-2\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial A}{\partial\theta}\right)\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial B}{\partial\theta}\right)+\frac{\partial^2 B}{\partial x^2}\left(\frac{1}{x^2}\frac{\partial^2 A}{\partial\theta^2}+\frac{1}{x}\frac{\partial A}{\partial x}\right), \tag{3b}$$

and (x, θ) are dimensionless plane polar coordinates.

The reciprocal load parameter

$$\epsilon^2 = \frac{D}{PR^2} \tag{4}$$

becomes small when the load P is large and the plate bending stiffness $D = Eh^3/[12(1-\nu^2)]$, where E is Young's modulus and ν is Poisson's ratio, is small. The solution obtained here is asymptotically valid as $\epsilon \to 0$.

If the parameter $\beta = 1$ then the equations apply to the plate and if $\beta = \lambda = h/H$ then they apply to the reinforcement.

Equation (2a) is the equation of transverse equilibrium in terms of the transverse displacement w, the nonlinear term on the right being the contribution from the membrane stresses due to the large deflection. The stress function f is chosen so that the membrane stress equilibrium equations are identically satisfied and f must then satisfy the compatibility equation (2b). The term on the right of (2b) arises from the stretching of the middle surface due to its transverse displacement w.

Dimensionless moments and membrane stress resultants are given by

$$M^{rr} = -\epsilon^2\beta^{-3}\left[\frac{\partial^2 w}{\partial x^2}+\nu\left(\frac{1}{x}\frac{\partial w}{\partial x}+\frac{1}{x^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right], \qquad M^{r\theta} = -\epsilon^2\beta^{-3}(1-\nu)\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial w}{\partial\theta}\right),$$

$$M^{\theta\theta} = -\epsilon^2\beta^{-3}\left[\frac{1}{x^2}\frac{\partial^2 w}{\partial\theta^2}+\frac{1}{x}\frac{\partial w}{\partial x}+\nu\frac{\partial^2 w}{\partial x^2}\right], \tag{5}$$

and

$$N^{rr} = \frac{1}{x^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{x}\frac{\partial f}{\partial x}, \qquad N^{r\theta} = -\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial f}{\partial \theta}\right), \qquad N^{\theta\theta} = \frac{\partial^2 f}{\partial x^2}. \tag{6}$$

Equations (5) are the moment-curvature constitutive relations and equations (6) define the stress function. We shall also require the equation for the component Q^r of the transverse shear acting on an edge of the plate whose normal is in the radial direction:

$$Q^r = -\epsilon^2 \beta^3 \frac{\partial}{\partial x}(\nabla^2 w). \tag{7}$$

Finally, in order to calculate the middle surface displacements u, v in the radial and circumferential directions respectively, we require the middle surface strain-displacement relations and the constitutive equations relating these strains to the membrane stress resultsnts:

$$\left.\begin{aligned} e^{rr} &= \frac{\partial u}{\partial x} + \epsilon^2\left(\frac{\partial w}{\partial x}\right)^2 = \beta(N^{rr} - \nu N^{\theta\theta}), \\ e^{r\theta} &= \frac{1}{2}\left(\frac{1}{x}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} - \frac{v}{x}\right) + \epsilon^2 \frac{1}{x}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta} = \beta(1+\nu)N^{r\theta}, \\ e^{\theta\theta} &= \frac{1}{x}\frac{\partial v}{\partial \theta} + \frac{u}{x} + \epsilon^2\left(\frac{1}{x}\frac{\partial w}{\partial \theta}\right)^2 = \beta(N^{\theta\theta} - \nu N^{rr}). \end{aligned}\right\} \tag{8}$$

The equations relating the dimensionless variables to the dimensional variables are listed in the Notation. The dimensionless variables are defined in such a way that, over most of the plate and reinforcement, they and their derivatives are order one. This means that, nearly everywhere, the right hand side of equation (2b), for example, is of the order ϵ^2 compared to the left hand side and may therefore be neglected in the first approximation, and similarly for the left hand side of equation (2a). In the case of equation (2a), however, neglecting the left hand side reduces the order of the equation by two with a corresponding decrease in the number of boundary conditions that can be imposed. What happens in this problem is that near the junction between the plate and the reinforcement (i.e. near $x = 1$) there is a bending boundary layer in which some of the derivatives on the left of (2a) become of order ϵ^{-2}, and these terms cannot be neglected in this region. This is typical of singular perturbation problems and the method of matched asymptotic expansions [9, 10] provides a mathematical formalism for dealing with this situation.

It is also important to remember that the displacements and strains refer to the middle surface of the plate (if $\beta = 1$) or reinforcement (if $\beta = \lambda$) and that the moments are moments about the middle surface and the membrane stress resultants act in the tangent plane to the middle surface. The form of the above equations is not affected by the eccentricity of the middle surfaces of the reinforcement and the plate; the factor β appears because variables in both regions are normalized relative to the plate thickness h. We shall take the origin of the coordinate system to lie in the plane of the middle surface of the undeformed plate so that the middle surface of the underformed reinforcement lies in the plane $Z = -(H-h)/2$, or in terms of the dimensionless coordinate system

$$z = -\delta = -(1-\lambda)/(2\lambda), \tag{9}$$

where we shall call δ the eccentricity. This eccentricity plays a vital role in formulating the boundary conditions at the junction between the plate and the reinforcement as we shall now see.

2.2 *Boundary conditions*

First, the boundary conditions at the junction of the reinforcement and the plate. In order to avoid introducing further sub- or superscripts we shall write all these boundary conditions in the form

$$\text{(reinforcement)} = \text{(plate)}.$$

That is, all quantities on the left of the equation are to be evaluated in the reinforcement and those on the right are to be evaluated in the plate, no other distinguishing notation will be used. Thus, at $x = 1$ we must have continuity of displacement and middle surface slope

$$u - \epsilon^2(2\delta\gamma)\frac{\partial w}{\partial x} = u, \qquad v - \epsilon^2(2\delta\gamma)\frac{1}{x}\frac{\partial w}{\partial \theta} = v, \tag{10}$$

where

$$\gamma = [6(1-\nu^2)]^{1/2}$$

and

$$w = w, \qquad \frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}, \tag{11}$$

continuity of two membrane stress components and transverse effective shear

$$N^{rr} = N^{rr}, \qquad N^{r\theta} = N^{r\theta}, \tag{12}$$

$$\gamma^{-1}Q_{\text{eff}} - \frac{\delta}{x}\frac{\partial N^{r\theta}}{\partial \theta} = \gamma^{-1}Q_{\text{eff}}, \tag{13a}$$

where

$$Q_{\text{eff}} \equiv N^{rr}\frac{\partial w}{\partial x} + N^{r\theta}\frac{1}{x}\frac{\partial w}{\partial \theta} + Q^r + \frac{1}{x}\frac{\partial M^{r\theta}}{\partial \theta}, \tag{13b}$$

and moment balance

$$M^{rr} - (\delta\gamma)N^{rr} = M^{rr}. \tag{14}$$

The last terms on the left hand sides of equations (10), (13a) and (14) arise due to the ecentricity of the reinforcement. The physical interpretation of these terms in (10) and (14) is clear, but the presence of an additional term in the effective shear, on the left of (13a), is not so obvious. It is explained if we note that the shear stress resultant $N^{r\theta}$ acting in the reinforcement, when transferred to the level of the plate middle surface, gives rise to a moment $-\delta N^{r\theta}$ about a radial axis. By the same argument used in explaining the Kirchhoff effective shear stress resultant, this moment is statically equivalent to a shear stress resultant of $-(\delta/x)(\partial N^{r\theta}/\partial\theta)$ on edge $x = \text{constant}$ in the reinforcement. Thus the effective shear on the left of equation (13a) must be modified by the addition of this quantity.†

†Additional terms such as those on the left sides of equations (10), (13a) and (14) arise whenever two plates are joined edge to edge with eccentricity between their middle surfaces. See, e.g. the discussion following equation (42) of [11].

Second, we must specify boundary conditions far from the hole, i.e. as $x \to \infty$

$$
\begin{aligned}
N^{rr} &\to \frac{1}{2}(1+\alpha)+\frac{1}{2}(1-\alpha)\cos 2\theta, \\
N^{r\theta} &\to \frac{1}{2}(1-\alpha)\sin 2\theta, \\
N^{\theta\theta} &\to \frac{1}{2}(1+\alpha)-\frac{1}{2}(1-\alpha)\cos 2\theta,
\end{aligned} \tag{15}
$$

where the right hand sides above are the dimensionless, polar coordinate transformation of equation (1). Also the plate remains flat at infinity so that

$$w(x,\theta)\to 0, \qquad \text{as} \qquad x\to\infty. \tag{16}$$

This last condition implies that the moments and effective transverse shear also vanish at infinity.

Finally, we take the inner edge of the reinforcement to be stress free, so that at $x = \rho$,

$$N^{rr}=0, \qquad N^{r\theta}=0, \qquad M^{rr}=0, \qquad Q_{\text{eff}}=0. \tag{17}$$

We shall now use the method of matched asymptotic expansions to find a solution of the above equations and boundary conditions that is asymptotically valid as $\epsilon \to 0$. In Section 3 we obtain outer expansions valid everywhere except in the boundary layer near $x = 1$ and in Section 4 we obtain inner expansions valid in the boundary layer. The matching of the inner and outer expansions is also carried out in Section 4 and in Section 4.4 we complete the solution by applying the boundary conditions at the junction $x = 1$.

3. THE OUTER SOLUTION

Outside the boundary layer we expand all variables as power series in ϵ. For example,

$$w(x,\theta;\epsilon)=\sum_{n=0}^{\infty}\epsilon^n w_n(x,\theta).$$

On substituting these expansions into equations (2) and into the boundary conditions (15) and (16), at infinity, and (17) at the inner edge of the reinforcement, and equating the coefficients of powers on ϵ on each side of the resulting equations we obtain the following system of equations: From equation (2a)

$$L(w_0,f_0)=0, \tag{18a}$$

$$L(w_1,f_0)=-L(w_0,f_1), \tag{18b}$$

$$L(w_2,f_0)=\beta^{-3}\nabla^4 w_0-L(w_0,f_2)-L(w_1,f_1), \tag{18c}$$

$$L(w_3,f_0)=\beta^{-3}\nabla^4 w_1-L(w_0,f_3)-L(w_1,f_2)-L(w_2,f_1),$$

etc.

and from equation (2b)

$$\nabla^4 f_0 = 0, \tag{19a}$$

$$\nabla^4 f_1 = 0,$$

$$\nabla^4 f_2 = -\beta^{-1} L(w_0, w_0), \tag{19c}$$

$$\nabla^4 f_3 = -\beta^{-1} 2L(w_0, w_1),$$

etc.

The boundary conditions in the plate at infinity are that as $x \to \infty$,

$$\left.\begin{aligned} N_0^{rr} &\to \frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)\cos 2\theta, \\ N_0^{r\theta} &\to -\frac{1}{2}(1-\alpha)\sin 2\theta, \\ N_0^{\theta\theta} &\to \frac{1}{2}(1+\alpha) - \frac{1}{2}(1-\alpha)\cos 2\theta, \end{aligned}\right\} \tag{20a}$$

$$\{N_n^{rr}, N_n^{r\theta}, N_n^{\theta\theta}\} \to 0, \qquad n \geq 1, \tag{20b}$$

and

$$w_n \to 0, \qquad n \geq 0. \tag{20c}$$

At the inner edge of the reinforcement at $x = \rho$

$$N_n^{rr} = N_n^{r\theta} = 0, \qquad n \geq 0, \tag{21a}$$

and

$$M_n^{rr} = 0, \qquad (Q_{\text{eff}})_n = 0, \qquad n \geq 0. \tag{21b}$$

Notice that, although equations (2) are nonlinear, equations (18) and (19), when solved sequentially, are linear. We first solve (19a, b) for f_0 and f_1. Next (18a), which is now linear since f_0 is known, is solved for w_0 and then (18b) can be solved for w_1. Equations (19c, d) are now solved for f_2 and f_3 since their right hand sides are now functions of the known variables w_0 and w_1. The procedure can be carried out to any desired order since once f_0 is determined the left hand sides of equations (18) become linear differential operators acting on the w_n and, at any step in the process, the right hand sides of (18) and (19) are functions of only those w_n and f_n determined in earlier steps.

3.1 *Outer solution in the plate*

In this problem we alter the above procedure slightly, for we observe that equations (18) have the trivial solution

$$w_n \equiv \text{constant} = 0, \quad n \geq 0 \tag{22}$$

which also satisfies the boundary condition (20c). Intuitively we expect that the plate will remain flat over most of its area, and so we take (22) as the solution we seek.

Equations (19) now all reduce to homogeneous, biharmonic equations

$$\nabla^4 f_n = 0, \qquad n \geq 0, \tag{23}$$

whose solutions, satisfying the boundary conditions (20a, b), are

$$f_0 = \frac{1}{4}(1+\alpha)x^2 + a_0 \ln x + \left[b_0 x^{-2} + c_0 - \frac{1}{4}(1-\alpha)x^2\right]\cos 2\theta, \tag{24a)†$$

$$f_n = a_n \ln x + (b_n x^{-2} + c_n)\cos 2\theta, \qquad n \geq 1. \tag{24b)†$$

The constants a_n, b_n, c_n, $n = 0, 1, 2, \ldots$, are determined by matching these solutions with the inner solutions, which in turn must satisfy the boundary conditions at the junction $x = 1$. This is explained in Section 4.

The terms in the expansions of the membrane stress resultants, which we obtain by substituting the results (24) into equations (6), are

$$\left.\begin{aligned} N_0^{rr} &= \frac{1}{2}(1+\alpha) + a_0 x^{-2} - \left[6b_0 x^{-4} + 4c_0 x^{-2} - \frac{1}{2}(1-\alpha)\right]\cos 2\theta, \\ N_0^{r\theta} &= -\left[6b_0 x^{-4} + 2c_0 x^{-2} + \frac{1}{2}(1-\alpha)\right]\sin 2\theta, \\ N_0^{\theta\theta} &= \frac{1}{2}(1+\alpha) - a_0 x^{-2} + \left[6b_0 x^{-4} - \frac{1}{2}(1-\alpha)\right]\cos 2\theta, \end{aligned}\right\} \tag{25a}$$

and

$$\left.\begin{aligned} N_n^{rr} &= a_n x^{-2} - (6b_n x^{-4} + 4c_n x^{-2})\cos 2\theta, \\ N_n^{r\theta} &= -(6b_n x^{-4} + 2c_n x^{-2})\sin 2\theta, \\ N_n^{\theta\theta} &= -a_n x^{-2} + 6b_n x^{-4}\cos 2\theta, \end{aligned}\right\} \tag{25b}$$

$n \geq 1$.

The displacement field can be determined from equations (8) by substituting the expansions, equating coefficients of powers of ϵ and integrating the resulting system of equations. We note that, since $w \equiv 0$ in the outer region, equations (8) are linear. As we do not use this result explicitly in this paper we omit it.

3.2 *Outer solution in the reinforcement*

Again we adopt the simplest solution to equations (18),

$$w_n(x, \theta) \equiv \text{constant} = \Omega_n, \text{ say}, \qquad n \geq 0. \tag{26}$$

It is not intuitively obvious that this solution is physically appropriate since, as the inner edge, $x = \rho$, of the reinforcement is free, we may expect it to wrinkle. However, we shall see that this solution is capable of satisfying the boundary and matching

†For solutions of the biharmonic equation in the theory of plane stress see, e.g. Timoshenko and Goodier[12].

conditions of the problem. It clearly satisfies the free edge boundary conditions (21b) of zero moment and effective shear.

As above, equations (19) reduce to $\nabla^4 f_n = 0$, $n = 0, 1, 2, \ldots$, and the solutions which satisfy the stress free boundary conditions (21a) at $x = \rho$ are

$$f_n = \bar{a}_n[\ln x - \tfrac{1}{2}x^2\rho^{-2}] + \{\bar{b}_n(x^{-2} - 3x^2\rho^{-4} + 2x^4\rho^{-6}) + \bar{c}_n(1 - 2x^2\rho^{-2} + x^4\rho^{-4})\}\cos 2\theta, \qquad n \geq 0. \tag{27}$$

Stress resultants, strains, and displacements are calculated as explained above. The constants $\bar{a}_n$, $\bar{b}_n$ and $\bar{c}_n$ are determined by matching with the inner solution in the reinforcement near $x = 1$.

4. THE INNER SOLUTION

4.1 *The boundary layer equations*

Near $x = 1$ we expect that w will be changing rapidly and that its derivatives with respect to x may be much greater than order one, in which case some of the terms in $\nabla^4 w$ and $L(w, w)$ in equations (2a) and (2b), respectively, will be very large. We therefore introduce the coordinate stretching transformation

$$y = (x - 1)/\epsilon, \tag{28}$$

which makes $\partial w/\partial y$ order one in the boundary layer if $\partial w/\partial x$ is order ϵ^{-1} there. The boundary layer equations are obtained in a more convenient form if the right hand side of equation (2a) is expressed in the alternative form (see equations (3b) and (6))

$$L(w, f) = N^{rr}\frac{\partial^2 w}{\partial x^2} + 2N^{r\theta}\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial w}{\partial \theta}\right) + N^{\theta\theta}\left(\frac{1}{x^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{x}\frac{\partial w}{\partial x}\right). \tag{29}$$

To obtain the boundary layer equations we make the coordinate transformation (28) in the equations of Section 2, substitute inner expansions of the form

$$w = \sum_{n=0}^{\infty} \epsilon^n \hat{w}_n(y, \theta), \tag{30}$$

and then equate the coefficients of powers of ϵ on each side of the resulting equations. From equation (2a) we obtain

$$\frac{\partial^4 \hat{w}_0}{\partial y^4} - \beta^3 \hat{N}_0^{rr}\frac{\partial^2 \hat{w}_0}{\partial y^2} = 0, \tag{31a}$$

$$\frac{\partial^4 \hat{w}_1}{\partial y^4} - \beta^3 \hat{N}_0^{rr}\frac{\partial^2 \hat{w}_1}{\partial y^2} = -2\frac{\partial^3 \hat{w}_0}{\partial y^3} + \beta^3 \hat{N}_1^{rr}\frac{\partial^2 \hat{w}_0}{\partial y^2} + \beta^3 2\hat{N}_0^{r\theta}\frac{\partial^2 \hat{w}_0}{\partial y \partial \theta} + \beta^3 \hat{N}_0^{\theta\theta}\frac{\partial \hat{w}_0}{\partial y}, \tag{31b}$$

etc.

and from (2b)

$$\begin{aligned} &\frac{\partial^4 \hat{f}_0}{\partial y^4} = 0, \\ &\frac{\partial^4 \hat{f}_1}{\partial y^4} = -2\frac{\partial^3 \hat{f}_0}{\partial y^3}, \\ &\frac{\partial^4 \hat{f}_2}{\partial y^4} = -2\frac{\partial^3 \hat{f}_1}{\partial y^3} + \frac{\partial^3 \hat{f}_0}{\partial y^3} - 2\frac{\partial^4 \hat{f}_0}{\partial y^2 \partial \theta^2} + \frac{\partial^2 \hat{f}_0}{\partial y^2}, \end{aligned} \tag{32a}$$

etc.

The terms of the inner expansions of the moments and the membrane stress resultants, obtained from equations (5) and (6), respectively, are

$$\hat{M}_0^{rr} = -\beta^{-3}\frac{\partial^2 \hat{w}_0}{\partial y^2}, \qquad \hat{M}_1^{rr} = -\beta^{-3}\left(\frac{\partial^2 \hat{w}_1}{\partial y^2} + \nu\frac{\partial \hat{w}_0}{\partial y}\right), \tag{33a}$$

$$\hat{M}_0^{r\theta} = 0, \qquad \hat{M}_1^{r\theta} = -\beta^{-3}(1-\nu)\frac{\partial^2 \hat{w}_0}{\partial y \partial \theta}, \tag{33b}$$

$$\hat{M}_0^{\theta\theta} = -\beta^{-3}\nu\frac{\partial^2 \hat{w}_0}{\partial y^2}, \qquad \hat{M}_1^{\theta\theta} = -\beta^{-3}\left(\nu\frac{\partial^2 \hat{w}_1}{\partial y^2} + \frac{\partial \hat{w}_0}{\partial y}\right), \tag{33c}$$

and

$$\left.\begin{aligned} \hat{N}_0^{rr} &= -y\frac{\partial \hat{f}_0}{\partial y} + \frac{\partial^2 \hat{f}_0}{\partial \theta^2} + \frac{\partial \hat{f}_1}{\partial y}, \\ \hat{N}_1^{rr} &= y^2\frac{\partial \hat{f}_0}{\partial y} - 2y\frac{\partial^2 \hat{f}_0}{\partial \theta^2} - y\frac{\partial \hat{f}_1}{\partial y} + \frac{\partial^2 \hat{f}_1}{\partial \theta^2} + \frac{\partial \hat{f}_2}{\partial y}, \end{aligned}\right\} \tag{34a}$$

$$\left.\begin{aligned} \hat{N}_0^{r\theta} &= y\frac{\partial^2 \hat{f}_0}{\partial y \partial \theta} + \frac{\partial \hat{f}_0}{\partial \theta} - \frac{\partial^2 \hat{f}_1}{\partial y \partial \theta}, \\ \hat{N}_1^{r\theta} &= -y^2\frac{\partial^2 \hat{f}_0}{\partial y \partial \theta} - 2y\frac{\partial \hat{f}_0}{\partial \theta} + y\frac{\partial^2 \hat{f}_1}{\partial y \partial \theta} + \frac{\partial \hat{f}_1}{\partial \theta} - \frac{\partial^2 \hat{f}_2}{\partial y \partial \theta}, \end{aligned}\right\} \tag{34b}$$

$$\hat{N}_0^{\theta\theta} = \frac{\partial^2 \hat{f}_2}{\partial y^2}, \qquad \hat{N}_1^{\theta\theta} = \frac{\partial^2 \hat{f}_3}{\partial y^2}. \tag{34c}$$

Note that in the plate $\beta = 1$, $y \geq 0$, and in the reinforcement $\beta = \lambda$, $y \leq 0$. Expressions for the terms in the inner expansions of the other variables can be obtained in a similar manner.

The boundary layer solutions must satisfy the boundary conditions at the junction between the plate and the reinforcement at $x = 1$. In terms of the inner expansion variables, and again using the convention (reinforcement) = (plate) of Section 2.2, these conditions are, from equations (10) and (11) for continuity of displacement and middle surface slope,

$$\hat{u}_0 = \hat{u}_0, \qquad \hat{u}_1 - 2\delta\gamma\frac{\partial \hat{w}_0}{\partial y} = \hat{u}_1, \qquad \text{etc.}, \tag{35a}$$

$$\hat{v}_0 = \hat{v}_0, \qquad \hat{v}_1 - 2\delta\gamma\frac{\partial \hat{w}_0}{\partial \theta} = \hat{v}_1, \qquad \text{etc.}, \tag{35b}$$

$$\hat{w}_n = \hat{w}_n, \qquad \frac{\partial \hat{w}_n}{\partial y} = \frac{\partial \hat{w}_n}{\partial y}, \qquad n \geq 0, \tag{36}$$

and continuity of stress resultants, equations (12),

$$(\hat{N}_n^{rr}, \hat{N}_n^{r\theta}) = (\hat{N}_n^{rr}, \hat{N}_n^{r\theta}), \qquad n \geq 0, \tag{37}$$

and effective shear (13),

$$\frac{\partial^3 \hat{w}_0}{\partial y^3} - \lambda^3 \hat{N}_0^{rr}\frac{\partial \hat{w}_0}{\partial y} = \frac{\partial^3 \hat{w}_0}{\partial y^3} - \hat{N}_0^{rr}\frac{\partial \hat{w}_0}{\partial y}, \tag{38a}$$

$$(\hat{Q}_{\text{eff}})_0 - \delta\gamma\frac{\partial \hat{N}_0^{r\theta}}{\partial \theta} = (\hat{Q}_{\text{eff}})_0, \qquad \text{etc.}, \tag{38b}$$

where

$$(\hat{Q}_{eff})_0 = -\beta^{-3}\left(\frac{\partial^2 \hat{w}_1}{\partial y^3} - \beta^3 \hat{N}_0^{rr}\frac{\partial \hat{w}_1}{\partial y}\right) - \beta^{-3}\frac{\partial^2 \hat{w}_0}{\partial y^2} + \hat{N}_1^{rr}\frac{\partial \hat{w}_0}{\partial y} + \hat{N}_0^{r\theta}\frac{\partial \hat{w}_0}{\partial \theta}, \tag{38c}$$

and lastly the moment balance equation (14) becomes

$$\hat{M}_0^{rr} - \delta\gamma\hat{N}_0^{rr} = \hat{M}_0^{rr}, \qquad \text{etc.} \tag{39}$$

At first glance, it might appear to be impossible to carry out the sequential solution of these equations since, for example, the order one term in the inner expansion for the membrane stress resultants depends on $\hat{f}_2$ (see equation 34c), and the order one term in the inner expansion for effective shear, equation (38c), involves $\hat{w}_1$ and $\hat{N}_1^{rr}$, which in turn involves $\hat{f}_2$ (see equations 31b and 34a). However, as we shall see next, the solution can be carried out sequentially mainly because equations (32a, b, c) for $\hat{f}_0$, $\hat{f}_1$ and $\hat{f}_2$ do not involve the transverse displacement. (The next equation in this sequence involves $\hat{w}_0$ but we shall not require $\hat{f}_3$.) Since the boundary layer equations are so complicated, only the leading term of the expansions for the moments and stress resultants have been found. In particular, we shall see that to order one the membrane stress resultants are not affected by the bending boundary layer.

4.2 *Solution of the boundary layer equations in the plate* ($y \geq 0$, $\beta = 1$)

The general solution of equation (32a) is

$$\hat{f}_0(\theta, y) = \phi_0(\theta) + \phi_1(\theta)y + \phi_2(\theta)y^2 + \phi_3(\theta)y^3, \tag{40}$$

where the functions $\phi_n(\theta)$, $n = 0, 1, 2, 3$, are to be determined from the boundary conditions at $y = 0$ and the matching condition.

In this problem it is necessary to carry out the matching of the inner and outer solutions before applying the boundary conditions at $y = 0$. Following Cole[8], we assume a region near $y = 0$, of overlapping validity of the inner solution (40) and the outer solution (24a) and introduce an intermediate variable

$$\bar{y} = \frac{(x-1)}{\mu(\epsilon)}, \qquad \text{so that} \qquad x = 1 + \mu\bar{y}, \qquad y = \frac{\mu}{\epsilon}\bar{y}, \tag{41a}$$

where

$$\mu(\epsilon) \to 0, \qquad \frac{\mu}{\epsilon} \to \infty, \qquad \text{as} \qquad \epsilon \to 0. \tag{41b}$$

Substituting from (41a) for x and y in $f_0(x, \theta)$ and $\hat{f}_0(y, \theta)$, respectively, and then expanding the result in powers of $\mu(\epsilon)$ we find

$$\begin{aligned} f_0 &= \tfrac{1}{4}(1+\alpha) + [b_0 + c_0 - \tfrac{1}{4}(1-\alpha)]\cos 2\theta \\ &\quad + \mu\bar{y}\{\tfrac{1}{2}(1+\alpha) + a_0 - [2b_0 + \tfrac{1}{2}(1-\alpha)]\cos 2\theta\} + 0(\mu^2), \end{aligned} \tag{42a}$$

$$\hat{f}_0 = \phi_0(\theta) + \left(\frac{\mu}{\epsilon}\right)\phi_1(\theta)\bar{y} + \left(\frac{\mu}{\epsilon}\right)^2\phi_2(\theta)\bar{y}^2 + \left(\frac{\mu}{\epsilon}\right)^3\phi_3(\theta)\bar{y}^3. \tag{42b}$$

Matching to order one consists in the agreement of the two expansions to terms of order one as $\epsilon \to 0$, $\bar{y}$ fixed, (i.e. $x \to 1$ and $y \to \infty$), or formally we have

$$\lim_{\substack{\epsilon \to 0 \\ \bar{y}\,\text{fixed}}} \{\hat{f}_0 + \epsilon \hat{f}_1 + \cdots - f_0 - \epsilon f_1 - \cdots\} = 0. \tag{43}$$

On substituting expansions (42) into (43) and performing the limit process indicated we see that

$$\phi_1(\theta) = \phi_2(\theta) = \phi_3(\theta) \equiv 0 \qquad \left(\text{since } \frac{\mu}{\epsilon} \to \infty \text{ as } \epsilon \to 0\right),$$

and

$$\hat{f}_0 = \phi_0(\theta) = \tfrac{1}{4}(1+\alpha) + [b_0 + c_0 - \tfrac{1}{4}(1-\alpha)] \cos 2\theta = f_0(\theta, 1). \tag{44}$$

Thus $\hat{f}_0$ is a function of θ alone and is just equal to the value of the outer solution $f_0(\theta, x)$ on the boundary $x = 1$. The constants a_0, b_0, c_0 remain to be determined from the boundary conditions.

We can now solve equation (32b) whose right hand side is zero and whose general solution is therefore

$$\hat{f}_1(\theta, y) = \psi_0(\theta) + \psi_1(\theta) y + \psi_2(\theta) y^2 + \psi_3(\theta) y^3. \tag{45}$$

Expanding this and the outer solution $f_1(\theta, x)$, equation (24b), in terms of theintermediate variable we have

$$\hat{f}_0 + \epsilon \hat{f}_1 = \phi_0(\theta) + \epsilon \psi_0(\theta) + \mu \psi_1(\theta) \bar{y} + \frac{\mu^2}{\epsilon} \psi_2(\theta) \bar{y}^2 + \frac{\mu^3}{\epsilon^2} \psi_3(\theta) \bar{y}^3, \tag{46a}$$

$$\begin{aligned} f_0 + \epsilon f_1 = {} & \tfrac{1}{4}(1+\alpha) + [b_0 + c_0 - \tfrac{1}{4}(1-\alpha)] \cos 2\theta \\ & + \mu \bar{y} \{\tfrac{1}{2}(1+\alpha) + a_0 - [2b_0 + \tfrac{1}{2}(1-\alpha)] \cos 2\theta\} \\ & + \epsilon (b_1 + c_1) \cos 2\theta + 0(\mu^2), \end{aligned} \tag{46b}$$

and matching to order ϵ consists in the agreement of the two expansions (46) to terms of order ϵ as $\epsilon \to 0$, $\bar{y}$ fixed, i.e.

$$\lim_{\substack{\epsilon \to 0 \\ \bar{y}\,\text{fixed}}} \left\{ \frac{\hat{f}_0 + \epsilon \hat{f}_1 + \cdots - f_0 - \epsilon f_1 - \cdots}{\epsilon} \right\} = 0.$$

Performing this limit process we find

$$\left. \begin{aligned} & \psi_0(\theta) = (b_1 + c_1) \cos 2\theta, \\ & \psi_1(\theta) = \tfrac{1}{2}(1+\alpha) + a_0 - [2b_0 + \tfrac{1}{2}(1-\alpha)] \cos 2\theta, \\ & \psi_2(\theta) = \psi_3(\theta) \equiv 0, \end{aligned} \right\} \tag{47}$$

and note that $\mu(\epsilon)$ has been further restricted since we have assumed that $\mu^2/\epsilon \to 0$ as $\epsilon \to 0$ so that $\epsilon \ll \mu \ll \sqrt{\epsilon}$. Substituting (47) into (45) gives us

$$\hat{f}_1(\theta, y) = (b_1 + c_1) \cos 2\theta + \{\tfrac{1}{2}(1+\alpha) + a_0 - [2b_0 + \tfrac{1}{2}(1-\alpha)] \cos 2\theta\} y. \tag{48}$$

In a similar manner we obtain

$$\hat{f}_2(\theta, y) = (b_2 + c_2)\cos 2\theta + (a_1 - 2b_1 \cos 2\theta)y + \{\tfrac{1}{4}(1+\alpha) - \tfrac{1}{2}a_0 + [3b_0 - \tfrac{1}{4}(1-\alpha)]\cos 2\theta)y^2, \quad (49)$$

the constants a_1, b_1, c_1, b_2, c_2 to be determined from the boundary conditions.

Equations (34) can now be used to obtain the terms in the inner expansion of the membrane stress resultants. In particular, the order one terms are

$$\left.\begin{aligned} \hat{N}_0^{rr}(\theta, y) &= \tfrac{1}{2}(1+\alpha) + a_0 + [\tfrac{1}{2}(1-\alpha) - 6b_0 - 4c_0]\cos 2\theta = N_0^{rr}(\theta, 1), \\ \hat{N}_0^{r\theta}(\theta, y) &= -[\tfrac{1}{2}(1-\alpha) + 6b_0 + 2c_0]\sin 2\theta = N_0^{r\theta}(\theta, 1), \\ \hat{N}_0^{\theta\theta}(\theta, y) &= \tfrac{1}{2}(1+\alpha) - a_0 - [\tfrac{1}{2}(1-\alpha) - 6b_0]\cos 2\theta = N_0^{\theta\theta}(\theta, 1) \end{aligned}\right\} \quad (50)$$

where the terms $N_0^{rr}(\theta, 1)$, etc. on the right are the leading terms of the outer expansion evaluated at the junction $x = 1$ between the plate and the reinforcement.

By expressing the stress-strain-displacement relation (8) in terms of the boundary layer variables, substituting the above expressions for the stresses and integrating the resulting equations, we can show that

$$\hat{u}_0(\theta, y) = u_0(\theta, 1), \qquad \hat{v}_0(\theta, y) = v_0(\theta, 1), \quad (51)$$

where the terms on the right are the leading terms in the outer expansions for the displacements (u, v) evaluated at the junction $x = 1$.

We now turn out attention to equations (31) to find the transverse displacement. Notice that $\hat{N}_0^{rr}$ which occurs as a coefficient on the left hand sides of these equations is a function of θ alone [see equations (50)] and so the general solution of equation (31a) is

$$\hat{w}_0(\theta, y) = A_0(\theta)\exp[-\sqrt{n(\theta)}y] + B_0(\theta)\exp[+\sqrt{n(\theta)}y] + C_0(\theta) + D_0(\theta)y, \quad (52a)$$

where

$$n(\theta) = \hat{N}_0^{rr}(\theta) = N_0^{rr}(\theta, 1). \quad (52b)$$

This must be matched to the outer solution in the plate which we recall is [equation (22)] $w_0(x, \theta) \equiv 0$ and it is clear that the formal matching procedure explained above is, in this case, equivalent to the condition

$$\hat{w}_0(\theta, y) \to 0 \text{ as } y \to \infty. \quad (53)$$

Hence, provided $n(\theta) > 0$, $0 \le \theta < 2\pi$, the result (52) reduces to

$$\begin{aligned} \hat{w}_o(\theta, y) &= A_0(\theta)\exp[-\sqrt{n(\theta)}y], \\ B_0(\theta) &= C_0(\theta) = D_0(\theta) \equiv 0, \end{aligned} \quad (54)$$

and $A_0(\theta)$ is determined from the boundary condition at $y = 0$.

Finally, the order one moments as found from equation (33) are

$$\hat{M}_0^{rr} = \nu^{-1}\hat{M}_0^{\theta\theta} = -A_0(\theta)n(\theta)\exp[-\sqrt{n(\theta)}y], \qquad \hat{M}_0^{r\theta} = 0. \tag{55}$$

4.3 *Solution of the boundary layer equations in the reinforcement* $(y \le 0, (y \le 0, \beta = \lambda = h/H)$

The solution of the boundary layer equations in the reinforcement is carried out as it was in the plate. Here we summarize the results. Having found the first three terms in the inner expansion of the stress function, we can calculate the order one membrane stress resultants which are

$$\begin{aligned}
\hat{N}_0^{rr}(\theta, y) &= -\bar{a}_0(1-\rho^2)\rho^{-2} - [6\bar{b}_0(1-\rho^{-4}) + 4\bar{c}_0(1-\rho^{-2})]\cos 2_\theta \\
&= N_0^{rr}(\theta, 1), \\
\hat{N}_0^{r\theta}(\theta, y) &= -[6\bar{b}_0(1+\rho^{-4}-2\rho^{-6}) + 2\bar{c}_0(1+2\rho^{-2}-3\rho^{-4})]\sin 2\theta \\
&= N_0^{r\theta}(\theta, 1), \\
\hat{N}_0^{\theta\theta}(\theta, y) &= -\bar{a}_0(1+\rho^2)\rho^{-2} + [6\bar{b}_0(1-\rho^{-4}+4\rho^{-6}) - 4\bar{c}_0(\rho^{-2}-3\rho^{-4})]\cos 2\theta \\
&= N_0^{\theta\theta}(\theta, 1),
\end{aligned} \tag{56}$$

where the terms on the right of each equation are the leading terms in outer expansions evaluated at $x = 1$. It can also be shown that the leading terms in the inner expansions of the displacements (u, v) are equal to the leading terms in the outer expansions evaluated at $x = 1$, that is

$$\hat{u}_0(\theta, y) = u_0(\theta, 1), \qquad \hat{v}_0(\theta, y) = v_o(\theta, 1). \tag{57}$$

It can now be seen that the solutions (50) and (51) in the plate, and (56) and (57) in the reinforcement, which must statisfy the boundary conditions (35a, b) and (37) in order to fix a_0, b_0, c_0 and $\bar{a}_0$, $\bar{b}_0$, $\bar{c}_0$, are not affected by the transverse displacement. Thus, to order one, the membrane stress resultants and displacements are the same as those for a plate reinforced symmetrically about its middle surface. Since the solution of this problem is well decumented in [1] and [13] we will not take it further here.

The leading term in the inner expansion for the transverse displacement is

$$\hat{w}_0(\theta, y) = \bar{B}_0(\theta)\exp[+\sqrt{\lambda^3 n(\theta)}y] + \Omega_0, \tag{58a}$$

where $n(\theta)$ is, by the boundary condition (37), as defined by equation (52b). That is

$$\hat{N}_0^{rr}(\text{plate}) = n(\theta) = \hat{N}_0^{rr}(\text{reinforcement}). \tag{58b}$$

The results (58a) satisfies the matching condition [cf. equation (26)] $w_0 \to \Omega_0$, as $y \to -\infty$, provided $n(\theta) > 0$, $0 \le \theta < 2\pi$.

The leading terms in the inner expansions of the bending moments are

$$\hat{M}_0^{rr} = \nu^{-1}\hat{M}_0^{\theta\theta} = -\bar{B}_0(\theta)n(\theta)\exp[+\sqrt{\lambda^3 n(\theta)}y], \qquad \hat{M}_0^{r\theta} = 0. \tag{59}$$

$\bar{B}_0(\theta)$ and Ω_0 remain with $A_0(\theta)$ [see equations (54) and (55)] to be determined from the boundary condition at $y = 0$ $(x = 1)$.

4.4 *Application of the boundary condition at the junction between the plate and reinforcement*
Continuity of transverse displacement, and middle surface slope, equations (36), gives

$$A_0(\theta) = \bar{B}_0(\theta) + \Omega_0, \qquad \text{and} \qquad A_0(\theta) = -\lambda^{3/2}\bar{B}_0(\theta).$$

Hence

$$A_0(\theta) = \lambda^{3/2}(1+\lambda^{3/2})^{-1}\Omega_0, \tag{60a}$$

$$\bar{B}_0(\theta) = -(1+\lambda^{3/2})^{-1}\Omega_0. \tag{60b}$$

Thus, the moment in the plate is, from equation (55),

$$\hat{M}_0^{rr} = -\lambda^{3/2}(1+\lambda^{3/2})^{-1}\Omega_0 n(\theta)\exp[-\sqrt{n(\theta)}y], \quad y \geq 0, \tag{61a}$$

and in the reinforcement, from equation (59),

$$\hat{M}_0^{rr} = (1+\lambda^{3/2})^{-1}\Omega_0 n(\theta)\exp[+\sqrt{\lambda^3 n(\theta)}y], \qquad y \leq 0. \tag{61b}$$

Entering these results into the moment balance boundary condition (39) we obtain

$$\Omega_0 = \delta\gamma, \tag{62a}$$

or expressed in terms of dimensional quantities, we find that outside the boundary layer the reinforcement has, to order one, a constant transverse displacement of

$$\bar{w}(\theta,r) \sim \frac{1}{2}(H-h). \tag{62b}$$

Thus, to order one, as $\epsilon \to 0$, the reinforcement is displaced so as to bring its middle surface into the plane of the middle surface of the plate provided the junction between the plate and reinforcement at $r = R$ is everywhere subjected to a radial tension.

We have still to satisfy the effective transverse shear boundary conditions (38). When (54) and (58a) are substituted into (38a) we see that both sides of (38a) are identically zero. Equation (38b) can be shown to be equivalent to the moment balance condition (39), and is also, therefore, satisfied by these results.

5. CONCLUDING REMARKS

The main reason for this investigation was to find what effect the eccentricity of the reinforcement has on the stress distribution around the hole. We have shown that, as a first approximation, with an error or order $\epsilon = [D/(PR^2)]^{1/2}$, the stresses are those obtained from the plane stress solution, except in a boundary layer region with a width of order ϵR on either side of the junction between the plate and the reinforcement at $r = R$. In this boundary layer, the plane stress solution stress are increased in the upper surface of the reinforcement and lower surface of the plate due to the bending moment induced by the eccentricity of the reinforcement. These stresses and bending moments have their maximum values at the junction $r = R$ where, in terms of the physical, dimensional variables of the problem, they are [see equations (55) and (59)]:

In the plate;

$$\bar{M}^{rr} = \nu^{-1}\bar{M}^{\theta\theta} \approx -\lambda^{3/2}(1+\lambda^{3/2})^{-1}\tfrac{1}{2}(H-h)\bar{n}(\theta), \tag{63a}$$

$$h\sigma/\bar{n}(\theta) \approx 3\lambda^{1/2}(1-\lambda)(1+\lambda^{3/2})^{-1}+1. \tag{63b}$$

In the reinforcement;

$$\bar{M}^{rr} = \nu^{-1}\bar{M}^{\theta\theta} \approx (1+\lambda^{3/2})^{-1}\tfrac{1}{2}(H-h)\bar{n}(\theta), \tag{64a}$$

$$h\sigma/\bar{n}(\theta) \approx 3\lambda(1-\lambda)(1+\lambda^{3/2})^{-1}+1 \tag{64b}$$

where $\lambda = h/H$, $\bar{n}(\theta)$ is the radial stress resultant at $r = R$, which must be positive (tension) for all θ if these results are to be valid, and σ is the radial extreme fibre stress.

The stress quotients (63b) and (64b) are shown as functions of λ in the graphs in Fig. 3. The

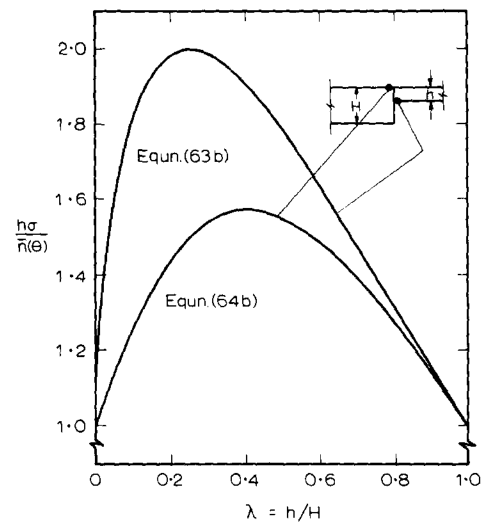

Fig. 3. Radial extreme fibre stress quotient vs thickness ratio λ.

severest case arises if $\lambda = 0{\cdot}25$ when the extreme fibre stress in the lower surface of the plate is exactly twice the stress that would occur in the symmetrically reinforced plate.

These results have a striking simplicity and we note that they depend only on the thicknesses of the plate and reinforcement, and on the radial tension at their junction. The result agrees with that of Wittrick[3] for the axisymmetric tension at infinity. Perhaps it is not unreasonable to conjecture that the results (63) and (64) are true for any shape of hole in a highly stretched plate

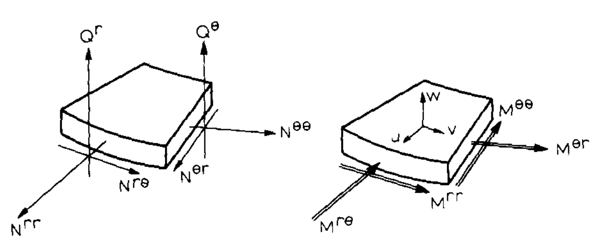

Fig. 4. Moments and stress resultants.

when the hole is reinforced by thickening the plate in the region of the hole on one side only. $\bar{n}(\theta)$ would, in this case, be the tension at the junction between the plate and the reinforcement in the direction normal to the junction curve, and calculated on the basis of the plane stress equations. Since a number of such plane stress solutions exist (see, e.g. Savin[1] and the references in Wittrick[2]), it is a simple matter to use (63) and (64) to estimate the change in the stress distribution at the junction due to eccentric reinforcement.

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